

“On Certain Definite Integrals. No. 13.” By W. H. L. RUSSELL, A.B., F.R.S. Received June 18, 1885.

In a paper which will be found in the “Proceedings of the Royal Society” for June, 1865, I gave methods for expressing the sum of certain series by definite integrals, or in other words, of expressing $F(x)$ by the form $\int PQ^x d\theta$. As shown in my last paper, this method is immediately connected with the solution of those partial differential equations which have constant coefficients by definite integrals, a circumstance which never crossed my mind till lately. In the present communication I hope to make further extensions in both these directions.

Case I. It was proved in the paper cited that the function

$$\sqrt[p]{\phi(n) + \sqrt[q]{\chi(n)}}$$

could be expressed in the form $\int PQ^n d\theta$, whereas $\phi(n)$ and $\chi(n)$ are rational (misprinted identical) functions of (n) . In the same way we may obtain $\sqrt[p]{\phi(n) + \sqrt[q]{\chi(n) + \sqrt[r]{\omega(n)}}}$. For it was proved in that paper that $\sqrt[p]{(\phi n + \sqrt[r]{\chi n})}$ can be expressed in the above form if $\epsilon^{\frac{1}{\chi(n)}}(\chi(n))^{\frac{s}{q}}$ can be thus expressed, and therefore

$$\sqrt[p]{\phi n + \sqrt[q]{(\chi(n) + \sqrt[r]{\omega(n)})}}$$

can be thus expressed in the form $\int PQ^r d\theta$ if

$$\epsilon^{\frac{1}{\chi(n) + \sqrt[r]{\omega n}}}(\chi(n) + \sqrt[r]{\omega(n)})^{\frac{s}{q}}$$

can be expressed in this form, which can be done by repeating the process.

This investigation assumes, however, that $\chi(n) + \sqrt[r]{\omega(n)}$ is less than unity.

Case II. Suppose it were required to reduce ϵ^N , where $N = \sqrt[p]{\phi(n) + \sqrt[q]{\chi(n) + \sqrt[r]{\omega(n)}}}$ to form $\int PQ^n d\theta$.

Then $\epsilon^N = \frac{1}{\pi} \int_0^{\pi} \frac{\epsilon^{\cos \theta} \cos(\sin \theta) (1 - N^2)}{1 - 2N \cos \theta + N^2} d\theta$, and since the denominator can be rationalised, we fall back on Case I. N must of course be less than unity.

Case III. When p is greater than 1

$$\frac{F}{p} = \frac{p^2 - 1}{\pi} \int_0^{\pi} \frac{F \epsilon^{\theta i} + F \epsilon^{-\theta i}}{1 - 2p \cos \theta + p^2} d\theta,$$

and $p^2 - 1 = p^2 - 2p \cos \theta + 1 + 2(p - \cos \theta) \cos \theta - 2 \sin^2 \theta$.

Hence

$$\begin{aligned} \frac{p^2-1}{1-2p\cos\theta+p^2} &= 1 + 2 \frac{(p-\cos\theta)}{(p-\cos\theta)^2+\sin^2\theta} \cdot \cos\theta \\ &\quad - \frac{2\sin^2\theta}{(p-\cos\theta)^2+\sin^2\theta} \\ &= 1 + 2 \cos\theta \int_0^\infty e^{-z(p-\cos\theta)} \cos z \sin\theta d\theta - 2 \sin\theta \int_0^\infty e^{-z(p-\cos\theta)} \sin z \sin\theta d\theta. \end{aligned}$$

By this means $F\left(\frac{1}{p}\right)$ can be expressed as double integral. So can $F(p)$, but then p must be less than unity.

We will now apply these considerations to the solution of linear partial differential equations.

$$\text{Let } F\left(\frac{d}{d\xi}, \frac{d}{d\eta}\right)u=0, \text{ or as we shall write it, } F\left(x\frac{d}{dx}, y\frac{d}{dy}\right)u=0,$$

then taking as before a specimen term Ax^my^n , m and n must be connected by the relations $F(m, n)=0$. Suppose from this we find

$$m = \sqrt[p]{\phi(n)} + \sqrt[q]{\chi(n)} + \sqrt[r]{\omega(n)} + \dots$$

Then, as will be seen by the reasoning employed in my former paper, the equation can be solved if

$$e^{\log x^p} \sqrt[p]{\phi(n)} + \sqrt[q]{\chi(n)} + \sqrt[r]{\omega(n)} + \dots$$

can be expressed in the form $fPQ^n d\theta$, which brings us to Case II.

The same process may in certain cases be applied to partial differential equations with three independent variables. Consider the series $A+Bx+B'y+Cx^2+C'xy+C''y^2+$, . . . when A, B, B' . . . are arbitrary constants. This may be written on Poisson's principles

$$F_1(x) + F_2x.y + F_3(x).y^2 + \dots$$

when F_1, F_2, F_3, \dots are arbitrary functions, and this again $F(x, y)$ when F is an arbitrary function of the two variables.

Now consider the partial differential equation $\frac{du}{d\xi} = 2 \frac{d^2u}{d\xi d\eta}$, or as I

shall write it $\left(z\frac{d}{dz}\right)u = 2\left(x\frac{d}{dx}\right)\left(y\frac{d}{dy}\right)u$, and let $Ax^my^nz^r$ be a specimen term of the solution, as in previous cases, then $r=2mn$, and our object must be to reduce $Ax^my^nz^{mn}$ to the form $fPQ_1^nQ_2^n$; this may be easily done by remembering that $2mn=(m+n)^2-m^2-n^2$, for

$$\int_{-\infty}^{\infty} e^{-(u-a)^2} du = \sqrt{\pi}$$

Hence

$$\int_{-\infty}^{\infty} e^{2au-u^2} du = e^{a^2} \sqrt{\pi}$$

and therefore

$$e^{(m+n)^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{2(m+n)u-u^2} du$$

also $e^{-m^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\rho^2} \cos 2m\rho d\rho$, and so for e^{-n^2} .

These transformations give the required form.

If we have two partial differential equations—

$$F_1\left(x\frac{d}{dx}, y\frac{d}{dy}, z\frac{d}{dz}\right)u=0,$$

$$F_2\left(x\frac{d}{dx}, y\frac{d}{dy}, z\frac{d}{dz}\right)u=0,$$

then substitute as before $\Delta x^m y^n z^r$ for u ; then we have the equations

$$F_1(m, n, r)=0, F_2(m, n, r)=0,$$

whence $m=\phi(r)$, $n=\chi(r)$, and we fall back on the first case.

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It follows from the expansion of $\cos^n \theta$ in terms of the cosines of the multiples of θ , that

$$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \dots \cdot \frac{n-r+1}{r} = \frac{2^n}{\pi} \int_0^\pi \cos n\theta \cos(n-2r)\theta d\theta,$$

and consequently this theorem can be used in the summation of series involving binomial coefficients. I propose to give a few examples of this.

From the binomial theorem, when the index is even, we have

$$\int_0^\pi d\theta \frac{\cos^{2n} \theta \sin(n-1)\theta \cos n\theta}{\sin \theta} = \frac{\pi}{2^{2n}} \left\{ 2^{2n-1} - 1 - 1 - \frac{(2n-1) \dots (n+1)}{1 \cdot 2 \dots (n-1)} \right\}$$

and when the index is odd,

$$\int_0^\pi d\theta \frac{\cos^{2n+1} \theta \sin n\theta \cos n\theta}{\sin \theta} = \pi \left\{ \frac{1}{2} - \frac{1}{2^{2n+1}} \right\}$$

Since $(1+x)^{n-1} = (1+x)^n (1-x+x^2-x^3 + \dots)$, therefore equating the coefficients of x^r , we have